SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

APRIL 2009
TASK #2
YEAR 12

Mathematics

General Instructions:

- Reading time—5 minutes.
- Working time—90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—71 Marks

- Attempt questions 1–6.
- The mark-value of each question is boxed in the right margin.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 3 sections:
  Section A(Questions 1 and 2),
  Section B(Questions 3 and 4),
  Section C(Questions 5 and 6),

Examiner: Mr P. Bigelow

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section A

Marks

Question 1 (12 marks)

(a) Find $\boxed{3}$

(i) $\displaystyle\int 2x^4\,dx,$

(ii) $\displaystyle\int \left(x^3 - 4x + 11\right)\,dx.$

(b) Find the sum of the first seven terms of the geometric series $48 + 24 + 12 + \ldots$ $\boxed{2}$

(c) A coin is tossed three times. $\boxed{3}$
Draw a tree diagram to show the possible outcomes.
What is the probability of

(i) three heads,

(ii) at least one tail,

(iii) two heads and one tail, in any order?

(d) Evaluate $\boxed{4}$

(i) $\displaystyle\int_1^9 \sqrt{x}\,dx,$

(ii) $\displaystyle\int_0^2 (1+x)^2\,dx.$

Marks

Question 2 (13 marks)

(a) Find $f''(x)$ if $f(x) = (x^2+1)^4$. $\boxed{2}$

(b) Given that 10, x and -1 are the first three terms of an arithmetic sequence, find: $\boxed{3}$

(i) x,

(ii) the common difference,

(iii) the next term.

(c) Find the equation of the normal to the curve $y = (2-3x)^2$ at the point where $x = 0$. $\boxed{3}$

(d) Find the curve, $y = f(x)$, for which $f'(x) = 6x^2 - 5x - 3$ and where $f(4) = 8$. $\boxed{3}$

(e) Evaluate $\displaystyle\sum_{r=1}^{\infty} 3.\left(\frac{1}{2}\right)^r$. $\boxed{2}$

Section B

(Use a separate writing booklet.)

Marks

Question 3 (12 marks)

(a) Given the table **2**

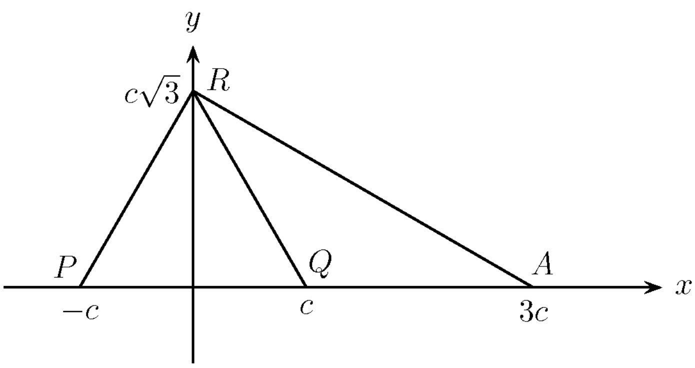
x	2	3	4	5	6
$f(x)$	2.1	3.2	4.1	2.9	1.7

use the Trapezoidal Rule with the above five values to approximate

$$\int_2^6 f(x)\,dx.$$

(b) Find the area bounded by $y = \sqrt{x}$, the y-axis and the lines $y = 2$ and $y = 3$. **2**

(c) **3**



(i) Show that $\triangle PQR$ is equilateral.

(ii) Show that $\triangle ARQ$ is isosceles.

(iii) Show that $RQ^2 = \frac{1}{3}AR^2$.

(d) A woman is employed at \$4600 per month, but at the end of each month she receives an increase of \$75. **3**

(i) How much will she earn during the twelfth month?

(ii) What will be her annual income?

(e) For what values of x is the curve $y = 6x^2 - x^3$ concave down? **2**

Marks

Question 4 (11 marks)

(a) A bag contains 3 Green, 5 Red and 2 White marbles. $\boxed{5}$

(i) One is chosen at random.
What is the probability that the marble is
(α) Red,

(β) Red or White?

(ii) Two are chosen (without replacement).
What is the probability that the marbles are
(α) both Red,

(β) the same colour,

(γ) Red and White?

(b) Given the curve $f(x) = 2x^3 - 12x^2$: $\boxed{4}$

(i) Find the stationary points and points of inflexion (establish their nature).

(ii) Sketch the curve, in the domain $-1 \leqslant x \leqslant 6$, showing all the essential features.

(c) The positive branch of the hyperbola $y = \dfrac{1}{x}$, between $x = 1$ and $x = 4$, is rotated about the x-axis. $\boxed{2}$
Find the volume of the solid of revolution, leaving your answer in exact form.

Section C

(Use a separate writing booklet.)

Marks

Question 5 (11 marks)

(a) A person borrows \$300 000 at an interest rate of 6% p.a., monthly reducible. At the end of each month he pays an installment of \$$M$. The loan is to be repaid in 30 years. The amount owing after n months is given by A_n. **7**

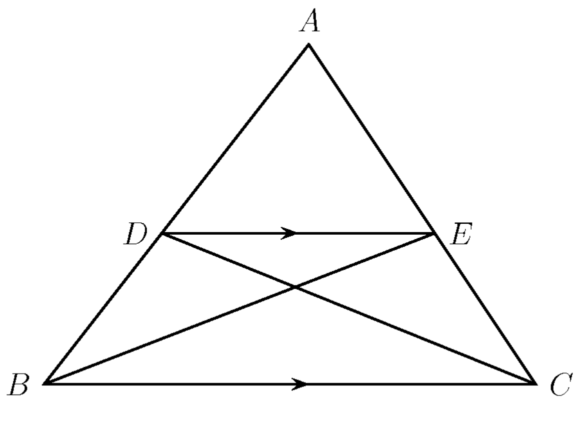
(i) Show that $A_1 = 300\,000 \times (1.005) - M$.

(ii) Find A_2.

(iii) Write down an expression for A_n.

(iv) Find M such that the loan is repaid in 30 years (give the answer to the nearest cent).

(b) **4**



$\triangle ABC$ is isosceles with $AB = AC$, and DE is parallel to BC.

(i) Prove that $DB = EC$.

(ii) Show that $\triangle BCD \equiv \triangle CBE$.

Marks

Question 6 (12 marks)

(a) A series is defined by $S_n = 2n^2 + 6n$ where S_n is the sum of the first n terms. $\boxed{4}$
Find

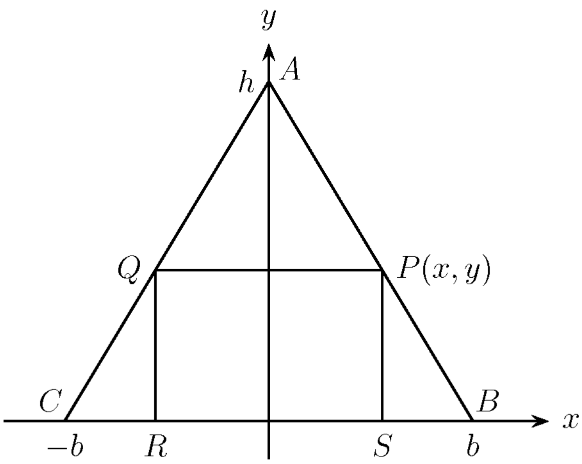
(i) S_{10},

(ii) t_n,

and

(iii) show that the series is arithmetic.

(b) $\boxed{8}$



ABC is an isosceles triangle where $AC = AB$. $PQRS$ is a rectangle inscribed in the triangle with P on AB.

(i) Show that $y = \dfrac{h}{b}(b - x)$.

(ii) Show that the area of rectangle $PQRS$ is $\mathbf{A} = \dfrac{2xh}{b}(b - x)$.

(iii) Find x such that area $\mathbf{A}$ is a maximum.

(iv) Show that this maximum area is half the area of the $\triangle ABC$.

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section A

Question 1:

a) i) $\int 2x^4\,dx$

$= \frac{2x^5}{5} + C$ (1)

$-\frac{1}{2}$ for no $+C$

ii) $\int (x^3 - 4x + 11)\,dx$

$= \frac{x^4}{4} - \frac{4x^2}{2} + 11x + C$

$= \frac{x^4}{4} - 2x^2 + 11x + C$ (2)

$-\frac{1}{2}$ for no $+C$

b) $r = \frac{24}{48} = \frac{12}{24} = \frac{1}{2}$

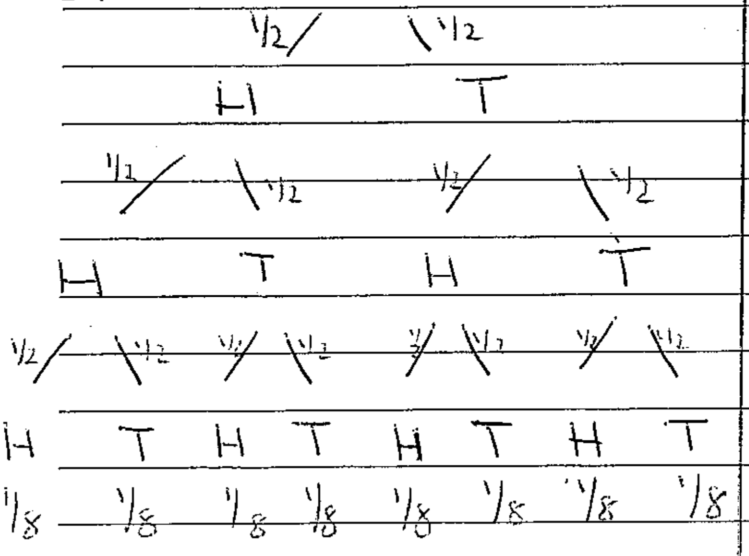
$a = 48$ ($\frac{1}{2}$)

$n = 7$ ($\frac{1}{2}$)

$S_7 = \frac{48(1 - \frac{1}{2}^7)}{1 - \frac{1}{2}}$

$= 95.25$ (1)

c)



i) $P(H\,H\,H) = \frac{1}{8}$ (1)

ii) $P(1 - HHH) = 1 - \frac{7}{8}$ (1)

iii) $P(HHT, HTH, THH)$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$= \frac{3}{8}$ (1)

d) i $\int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2}\,dx$

$= \left[\frac{x^{3/2}}{3/2}\right]_1^9$ (1)

$= \left[\frac{9^{3/2}}{3/2}\right] - \left[\frac{1^{3/2}}{3/2}\right]$

$= 18 - \frac{2}{3}$

$= 17\frac{1}{3}$ (1)

ii $\int_0^2 (1+x)^2\,dx$

$= \left[\frac{(1+x)^3}{3}\right]_0^2$ (1)

$= \left[\frac{3^3}{3}\right] - \left[\frac{1^3}{3}\right]$

$= 9 - \frac{1}{3}$

$= 8\frac{2}{3}$ (1)

Section A

Question 2:

a) $f(x) = (x^2+1)^4$

$f'(x) = 4(x^2+1)^3 \times 2x$

$= 8x(x^2+1)^3$ (1)

$\quad u \qquad v$

$f''(x) = uv' + vu'$

$= 8x \times 3(x^2+1)^2 \times 2x$

$+ (x^2+1)^3 \times 8$

$= 48x^2(x^2+1)^2 + 8(x^2+1)^3$

$= 8(x^2+1)^2\left[6x^2 + (x^2+1)^1\right]$

$= 8(x^2+1)^2\left[7x^2+1\right]$ (1)

b) i)

$x - 10 = -1 - x$

$2x = 9$

$x = 9/2$ (1)

ii) $d = 10 - 9/2$

$= 5\tfrac{1}{2}$ (1)

iii) $T_4 = -1 - 5\tfrac{1}{2}$

$= \frac{-13}{2}$ or $-6\tfrac{1}{2}$ (1)

c) $y = (2-3x)^2$

$y' = 2(2-3x)^1 \times -3$

$= -6(2-3x)$

when $x = 0$

$y' = -6(2-0)$

$y' = -12.$

gradient of normal $= \frac{-1}{-12}$

$= 1/12$ (1)

when $x = 0$, $y = (2-0)^2$

$= 4$ (1)

equation of normal

$y - 4 = 1/12(x-0)$

$12y - 48 = x$

$x - 12y + 48 = 0$ (1)

d) $f'(x) = 6x^2 - 5x - 3$

$f(x) = \frac{6x^3}{3} - \frac{5x^2}{2} - 3x + C$

$= 2x^3 - \frac{5}{2}x^2 - 3x + C$ (1)

$f(4) = 8$

$\therefore 8 = 2(4)^3 - \frac{5}{2}(4)^2 - 3(4) + C$

$8 = 128 - 40 - 12 + C$

$8 = 76 + C$

$C = -68$ (1)

$\therefore f(x) = 2x^3 - \frac{5x^2}{2} - 3x - 68$ (1)

e) $T_1 = 3/2$, $T_2 = 3/4$ $\therefore r = 1/2$

$a = 3/2$ (1/2), $r = 1/2$ (1/2)

$S_\infty = \frac{3/2}{1 - 1/2}$

$= 3$ (1)

SECTION B

3. $\frac{h}{2}\left\{f(2) + f(6) + 2\left[f(3) + f(4) + f(5)\right]\right\}$

(a) $h = \frac{b-a}{n} = \frac{6-2}{4} = 1$

$\frac{1}{2}\left\{3.8 + 2 \times 10.2\right\}$

$= 12.1$

(b) $x = y^2$

$A = \int_{2}^{3} y^2\, dy$

$= \left[\frac{y^3}{3}\right]_2^3 = 9 - \frac{8}{3}$

$A = 6\frac{1}{3} u^2$

(c) (i) $PR^2 = c^2 + (\sqrt{3}c)^2 = 4c^2 \quad PR = 2c$

$QR^2 = c^2 + (\sqrt{3}c)^2 = 4c^2 \quad QR = 2c$

$PQ = 2c$

$\triangle PQR$ is equilateral

(ii) $QA = 3c - c = 2c = QR$

$\triangle ARQ$ is isosceles

(iii) $AR^2 = (3c)^2 + (\sqrt{3}c)^2 = 12c^2$

$RQ^2 = 4c^2$ (from(i)) $= \frac{1}{3} AR^2$

(d)

(i) $4600 + 12 \times 75 = \$5500$

(ii) earnings $4675 + 4675 + 75 \; \ldots \; 5500$

arithmetic series $a = 4675 \quad n = 12 \quad l = 5500$

$S_{12} = \frac{n}{2}(a + l) = 6(4675 + 5500)$

$= \$61050$

(e) Concave down when $y'' < 0$

$y = 6x^2 - x^3$

$\frac{dy}{dx} = 12x - 3x^2 \qquad \frac{d^2y}{dx^2} = 12 - 6x$

$12 - 6x < 0 \qquad -6x < 12$

Curve concave down $x > 2$

SECTION B (cont)

4(a)

(α)(i) $P(red) = 5/10 = \frac{1}{2}$

(β) $P(R \text{ or } W) = 7/10$

(α)(ii) $P(RR) = \frac{5}{10} \times \frac{4}{9} = \frac{2}{9}$

(β) $P(RR + GG + WW)$

$\frac{2}{9} + \frac{3}{10} \times \frac{2}{9} + \frac{2}{10} \times \frac{1}{9} = \frac{14}{45}$

(γ) $P(RW + WR)$

$= \frac{5}{10} \times \frac{2}{9} + \frac{2}{10} \times \frac{5}{9} = \frac{2}{9}$

(b)

(i) $f'(x) = 6x^2 - 24x$

Stat pts $6x(x-4) = 0$ $\quad x = 0 \quad x = 4$

$y = 0 \quad y = -64$

at $x = -1$ $f'(x) > 0$

$x = 1$ $f'(x) < 0$ $\therefore$ max at $x = 0$

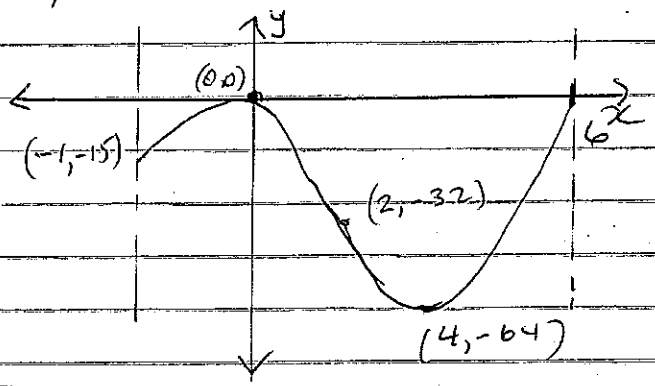
at $x = 3$ $f'(x) < 0$

$x = 5$ $f'(x) > 0$ $\therefore$ min at $x = 4$

$f''(x) = 12x - 24 = 0$ when $x = 2$

Point of inflexion at $(2, -32)$

(ii)



(c) $V = \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx$

$= \pi \left[-x^{-1}\right]_1^4$

$= \pi \left[-\frac{1}{4} - (-1)\right]$

$= \frac{3}{4}\pi \quad u^3$

11

Section C 2 Unit 2009 May exam

5 (a) \$300,000 6% p.a. monthly reducible $\Rightarrow \frac{6}{12} = 0.5\%$ per month

installment \$m.

repaid 30 years $\Rightarrow$ 360 months

(i) $A_1 = 300\,000 + 300\,000 \times 0.5\% - m$

$= 300,000(1 + 0.5\%) - m$

$= 300,000(1.005) - m$ 1

(ii) $A_2 = [300,000(1.005) - m] + [300,000(1.005) - m] \times 0.5\% - m$

$= [300,000(1.005) - m][1 + 0.5\%] - m$

$= [300,000(1.005) - m][1.005] - m$

$= 300,000(1.005)^2 - m - 1.005m$

$= 300\,000(1.005)^2 - m(1 + 1.005)$ 2

(iii) $\therefore A_n = 300,000(1.005)^n - m(1 + 1.005 + \dots + 1.005^{n-1})$ 1

(iv) $A_{360} = 300000(1.005)^{360} - m(1 + 1.005 + \dots + 1.005^{359})$

$m = \dfrac{300\,000(1.005)^{360}}{(1 + 1.005 + \dots + 1.005^{359})}$. using $S_n = \dfrac{rl - a}{r - 1}$

$= \dfrac{300\,000(1.005)^{360}}{\frac{1.005 \times 1.005^{359} - 1}{1.005 - 1}} = \dfrac{300\,000(1.005)^{360}}{\frac{(1.005)^{360} - 1}{.005}} = \$1798.65$ 3

5 (b) (i) $\hat{ADE} = \hat{ABC}$ alt. angles, $DE \parallel BC$.
$\hat{AED} = \hat{ACB}$ alt. angles, $DE \parallel BC$.

So given $\triangle ABC$ is isosceles, $AB = AC$
then $\triangle ADE$ is isosceles $AD = AE$.

$$\left.\begin{aligned} \text{So } AB &= AD + DB \\ AC &= AE + EC \end{aligned}\right\} DB = EC.$$ (2)

(ii) To prove $\triangle BCD \equiv \triangle CBE$
$BD = CE$ proven above.
$\hat{DBC} = \hat{ECB}$ isosceles $\triangle ABC$.
BC = common side
$\therefore \triangle BCD \equiv \triangle CBE$ (SAS) (2)

/12

6 (a) $S_n = 2n^2 + 6n$

(i) $S_{10} = 2\times 10^2 + 6\times 10 = 260$ 1

(ii) $t_n = S_n - S_{n-1}$

$= (2n^2+6n) - (2(n-1)^2 + 6(n-1))$

$= (\cancel{2n^2} + \cancel{6n}) - (\cancel{2n^2} - 4n + 2 + \cancel{6n} - 6)$

$t_n = 4n + 4$ 2

(iii) $t_1 = 4\times 1 + 4 = 8$
$t_2 = 4\times 2 + 4 = 12$
$t_3 = 4\times 3 + 4 = 16$

$a = 8$
$d = 4$
series is arithmetic 1

(b) (i) $A(0,h)$ $B(6,0)$

gradient $m = \frac{0-h}{6-0} = -\frac{h}{6}$

using $(y-y_1) = m(x-x_1)$ and $(6,0)$

$(y-0) = -\frac{h}{6}(x-6)$

$y = -\frac{h}{6}(x-6)$

$y = \frac{h}{6}(6-x)$. 2

(ii) area PQRS $= 2x \times y$

$= 2x \times \frac{h}{6}(6-x)$

$= \frac{2xh}{6}(6-x)$. 1

(iii) $A = \frac{2xh}{b}(b-x) = \frac{2xhb}{b} - \frac{2x^2h}{b} = 2xh - \frac{2x^2h}{b}$

$A' = 2h - \frac{4xh}{b}$

$A'' = -\frac{4h}{b}$

When $A' = 0$, $2h = \frac{4xh}{b}$

$4xh = 2hb$ ② ①

$x = \frac{2hb}{4h} = \frac{b}{2}$ $A'' < 0$ max

(iv) So max area is $A = 2xh - \frac{2x^2h}{b}$

$= 2 \times \frac{b}{2} \times h - \frac{2 \times \frac{b^2}{4} \times h}{b}$

$= bh - \frac{bh}{2} = \frac{bh}{2}$

area $\triangle ABC$

$= 2 \times \frac{1}{2} \times b \times h$

$= bh$ ②

max area PQRS $= \frac{bh}{2}$

area $\triangle ABC = bh$

} half the area of $\triangle ABC$